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2010 J. Phys. A: Math. Theor. 43 025304

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Weak values with decoherence

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Received 15 July 2009, in final form 7 November 2009

Published 10 December 2009

Online at stacks.iop.org/JPhysA/43/025304

Abstract

The weak value of an observable is experimentally accessible by weak measurements as theoretically analyzed by Aharonov *et al* and recently experimentally demonstrated. We introduce a W operator associated with the weak values and give a general framework of quantum operations to the W operator in parallel with the Kraus representation of the completely positive map for the density operator. The decoherence effect is also investigated in terms of the weak measurement by a shift of a probe wavefunction of continuous variable. As an application, we demonstrate how the geometric phase is affected by the bit flip noise.

PACS numbers: 03.65.Ta, 03.65.Ca, 03.65.Yz, 03.65.Vf

1. Introduction

The concept of measurement has been a longstanding problem in quantum mechanics. It was discussed by von Neumann [1] in the mathematical foundations of quantum mechanics and further developed by Kraus [2], Davies and Lewis [3], Ozawa [4] and other people (e.g. see [5, 6]). According to the extended Born rule, the positive operator valued measure (POVM) M_i gives the probability distribution $P_i = \text{Tr}[\rho M_i]$ in quantum measurement with ρ being the density operator corresponding to a prepared state. Measurement changes the quantum state as a positive map from a density operator $\rho (>0)$ to another density operator $\mathcal{E}(\rho) (>0)$ because of the probabilistic interpretation. The map \mathcal{E} is called a completely positive map (CP map) if the map remains positive when it is trivially extended to any larger Hilbert space. That is $(\mathcal{E} \otimes I)(\rho_{ex}) > 0$ for any state ρ_{ex} of an arbitrarily extended system when the map \mathcal{E} is completely positive. Physically, this is a very reasonable requirement because there should always exist the outside of an experimental setup which is inactive during the experiment procedures [4].

It is known that the seemingly humble requirement of the complete positivity of the quantum operation \mathcal{E} together with the trace preservation and the positive convexity for the density operator ρ implies an explicit representation of the physical operation in the Kraus form: $\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger$, where E_i is called the Kraus operator and $M_i := E_i^\dagger E_i$ is the POVM, with the property of the decomposition of unity, $\sum_i E_i^\dagger E_i = \sum_i M_i = 1$. The Kraus representation of physical operations is a powerful tool in quantum information theory [7].

However, the probability distribution is not the only thing that is experimentally accessible in quantum mechanics. In quantum mechanics, the phase is also an essential ingredient and, in particular, the geometric phase is a notable example of an experimentally accessible quantity [8]. The general experimentally accessible quantity which contains complete information of the probability and the phase seems to be the *weak value* advocated by Aharonov and his collaborators [9, 10]. They proposed a model of weakly coupled system and probe (see section 3.1.) to obtain information to a physical quantity as a ‘weak value’ only slightly disturbing the state. Here, we briefly review the formal aspects of the weak value.

Weak value: For an observable A , the weak value $\langle A \rangle_w$ is defined as

$$\langle A \rangle_w = \frac{\langle f | U(t_f, t) A U(t, t_i) | i \rangle}{\langle f | U(t_f, t_i) | i \rangle} \in \mathbb{C}, \quad (1)$$

where $|i\rangle$ and $\langle f|$ are normalized pre-selected ket and post-selected bra state vectors, respectively. Here, $U(t_2, t_1)$ is an evolution operator from the time t_1 to t_2 . The weak value $\langle A \rangle_w$ actually depends on the pre- and post-selected states $|i\rangle$ and $\langle f|$ but we omit them for notational simplicity in the case that we fix them. Otherwise, we write them explicitly as ${}_f \langle A \rangle_i^w$ instead of $\langle A \rangle_w$. The denominator is assumed to be non-vanishing. Note also that the weak value $\langle A \rangle_w$ is independent of the phases of the pre- and post-selected states so that it is defined in the ray space.

The physical intuition may be enhanced by looking at the identity for the expectation value of an observable A :

$$\begin{aligned} \langle i | U^\dagger(t, t_i) A U(t, t_i) | i \rangle &= \sum_f |\langle f | U(t_f, t_i) | i \rangle|^2 \frac{\langle f | U(t_f, t) A U(t, t_i) | i \rangle}{\langle f | U(t_f, t_i) | i \rangle} \\ &= \sum_f p_f \cdot {}_f \langle A \rangle_i^w, \end{aligned} \quad (2)$$

where $p_f = |\langle f | U(t_f, t_i) | i \rangle|^2$ is the probability to obtain the final state $\langle f|$ given the initial state $|i\rangle$ [11]. Comparing with the standard probability theory, one may interpret the weak value as a complex random variable with the probability measure p_f .³ The statistical average of the weak value coincides with the expectation value in quantum mechanics. Further, if an operator A is a projection operator $A = |a\rangle\langle a|$, the above identity becomes an analog of the Bayesian formula

$$|\langle a | U(t, t_i) | i \rangle|^2 = \sum_f p_f \cdot {}_f \langle |a\rangle\langle a| \rangle_i^w. \quad (3)$$

The left-hand side is the probability to obtain the state $|a\rangle$ given the initial state $|i\rangle$. From this, one may obtain some intuition by interpreting the weak value ${}_f \langle |a\rangle\langle a| \rangle_i^w$ as the (complex!) conditional probability of obtaining the result $|a\rangle$ under an initial condition $|i\rangle$ and a final condition $|f\rangle$ in the process $|i\rangle \rightarrow |a\rangle \rightarrow |f\rangle$.⁴ We believe that the concept of a quantum

³ In the standard probability theory [53], the expectation value of an observable A is given as a probabilistic average, $\langle A \rangle = \int dp h_A(p)$, with $h_A(p)$ and dp being the random variable associated with A and probability measure, which is independent of A , respectively. The standard expression of the expectation value, $\langle A \rangle = \sum_n a_n |\langle a_n | i \rangle|^2$, is given by the Born rule. However, this does not fit the standard probability theory because the probability measure depends on A .

⁴ The interpretation of the weak value as a complex probability is suggested in the literature [20].

trajectory can be formulated in the framework of weak values [12]. This interpretation of the weak values gives many possible examples of strange phenomena such as a negative kinetic energy [13], a spin $100\hbar$ for an electron [9] and a superluminal propagation of light [14]. Of course, we should not take the strange weak values too literally but the remarkable consistency of the framework of the weak values due to equation (3) and a consequence of the completeness relation

$$\sum_a \langle |a\rangle \langle a| \rangle_w = 1, \tag{4}$$

may give a useful concept to further push theoretical consideration by intuition. The framework of weak values has been theoretically applied to quantum stochastic process [15], the tunneling traverse time [16], non-locality and consistent history [17], semi classical weak values [18], counterfactual reasonings [19, 20] and quantum communications [21]. However, the most important fact is that the weak value is experimentally accessible by the weak measurement (e.g. see [22–29]) so that the intuitive argument based on the weak values can be either verified or falsified by experiments. Historically, the terminology ‘weak value’ comes from the weak measurement, where the coupling between the target system and the probe is weak, explained in section 3.1. Below, we introduce a formal concept of a W operator as a tool to formally describe the weak value.

W operator: We define a W operator $W(t)$ as

$$W(t) := U(t, t_i) |i\rangle \langle f| U(t_f, t). \tag{5}$$

To facilitate the formal development of the weak value, we introduce the ket state $|\psi(t)\rangle$ and the bra state $\langle\phi(t)|$ as

$$|\psi(t)\rangle = U(t, t_i) |i\rangle, \quad \langle\phi(t)| = \langle f| U(t_f, t), \tag{6}$$

so that the expression for the W operator simplifies to

$$W(t) = |\psi(t)\rangle \langle\phi(t)|. \tag{7}$$

By construction, the two states $|\psi(t)\rangle$ and $\langle\phi(t)|$ satisfy the Schrödinger equations with the same Hamiltonian with the initial and final conditions $|\psi(t_i)\rangle = |i\rangle$ and $\langle\phi(t_f)| = \langle f|$. In a sense, $|\psi(t)\rangle$ evolves forward in time while $\langle\phi(t)|$ evolves backward in time. The time reverse of the W operator (7) is $W^\dagger = |\phi(t)\rangle \langle\psi(t)|$. Thus, we can say that the W operator is based on the two-state vector formalism [30]. Historically speaking, the two-state vector formalism was originally motivated by the time symmetric description of quantum measurement [31] and has been related to the weak values and weak measurement [32] also developed by Aharonov *et al* [33, 34]. Even an apparently similar quantity to the W operator (7) was introduced by Reznik and Aharonov [35] in the name of ‘two-state’ with the conceptually different meaning. The W operator gives the weak value of the observable A as

$$\langle A \rangle_w = \frac{\text{Tr}(WA)}{\text{Tr}W}, \tag{8}$$

in parallel with the expectation value of the observable A by $\text{Tr}(\rho A)/\text{Tr}\rho$ from Born’s probabilistic interpretation. Furthermore, the W operator (5) can be regarded as a special case of a standard purification of the density operator [36]. In our opinion, the W operator should be considered on the same footing of the density operator. For a closed system, both satisfy the Schrödinger equation. In a sense, the W operator W is the square root of the density operator since

$$W(t)W^\dagger(t) = |\psi(t)\rangle \langle\psi(t)| = U(t, t_i) |i\rangle \langle i| U^\dagger(t, t_i), \tag{9}$$

which describes a state evolving forward in time for a given initial state $|\psi(t_i)\rangle\langle\psi(t_i)| = |i\rangle\langle i|$, while

$$W^\dagger(t)W(t) = |\phi(t)\rangle\langle\phi(t)| = U(t_f, t)|f\rangle\langle f|U^\dagger(t_f, t), \quad (10)$$

which describes a state evolving backward in time for a given final state $|\phi(t_f)\rangle\langle\phi(t_f)| = |f\rangle\langle f|$. The W operator describes the entire history of the state from the past (t_i) to the future (t_f) and measurement performed at the time t as we shall see in section 3.1. This description is conceptually different from the conventional one by the time evolution of the density operator.

Our aim is to find the most general map for the W operator W . The result turns out to be of the form $\mathcal{E}(W) = \sum_m E_m W F_m^\dagger$.⁵ The W operator is a useful tool to compactly describe the effect of decoherence to the weak values just as the density operator is to the expectation value in the standard theory of decoherence [37]. Furthermore, the amount of the effect due to the environment in the weak measurement is exactly given by the weak value defined by the quantum operation of the W operator $\mathcal{E}(W)$.

This paper is organized as follows. In section 2, we recapitulate the CP map as quantum operations for density operators. We define a W operator motivated by the two-state vector formalism and construct quantum operations for the W operator instead of the density operator. Because of the complete positivity of the quantum operation, we can construct the state change defined by operations solely on the target Hilbert space similarly to the Kraus representation. In section 3, we review the process to obtain weak values by the weak measurement [9, 38]. We analyze the target system with the noise during the weak measurement by the shifts of the probe observables. In section 4, we show an application of the quantum operation to the W operator. Since the geometric phase can be formally given by weak values [39], we operationally define the geometric phase for mixed states on the basis of the quantum operation of the W operator. Section 5 is devoted to the summary.

2. Quantum operations

2.1. Quantum operations for density operators—review

Let us recapitulate the general theory of quantum operations of a finite-dimensional quantum system [7, chapter 8]. All physically realizable quantum operations can be generally described by a CP map [4, 40], since the isolated system of a target system and an auxiliary system always undergoes the unitary evolution according to the axiom of quantum mechanics [1]. One of the important properties of the CP map is that all physically realizable quantum operations can be described only by operators defined in the target system.

Let \mathcal{E} be a positive map from $\mathcal{L}(\mathcal{H}_s)$, a set of linear operations on the Hilbert space \mathcal{H}_s , to $\mathcal{L}(\mathcal{H}_s)$. If \mathcal{E} is completely positive, its trivial extension σ from $\mathcal{L}(\mathcal{H}_s)$ to $\mathcal{L}(\mathcal{H}_s \otimes \mathcal{H}_e)$ is also positive such that

$$\sigma(|\alpha\rangle) := (\mathcal{E} \otimes I)(|\alpha\rangle\langle\alpha|) > 0, \quad (11)$$

for an arbitrary state $|\alpha\rangle \in \mathcal{H}_s \otimes \mathcal{H}_e$. We assume without loss of generality $\dim\mathcal{H}_s = \dim\mathcal{H}_e < \infty$. Throughout this paper, we concentrate on the case that the target state is pure though the generalization to mixed states is straightforward. From the complete positivity, we obtain the following theorem for quantum state changes.

⁵ Of course, if we introduce a probe Hilbert space and then consider the probability distribution of the measurement outcome, we can extract information of the phase by an interference pattern (e.g. see [54, 55]). The virtue of the CP map is that it is defined solely by operations of the target Hilbert space. We would like to find out a representation analogous to the Kraus representation for the phase related object in a way defined only in the target system. A special case of weak values in the environment has recently been discussed on the post-selected decay rate in [56].

Theorem 1. For any quantum state $|\psi\rangle_s \in \mathcal{H}_s$, we expand

$$|\psi\rangle_s = \sum_m \psi_m |m\rangle_s, \quad (12)$$

with a fixed complete orthonormal set $\{|m\rangle_s\}$. Then, a quantum state change can be written as

$$\mathcal{E}(|\psi\rangle_s \langle\psi|) = e \langle \tilde{\psi} | \sigma(|\alpha\rangle) | \tilde{\psi} \rangle_e, \quad (13)$$

where

$$|\tilde{\psi}\rangle_e := \sum_m \psi_m^* |m\rangle_e. \quad (14)$$

Here $|\alpha\rangle$ is a maximally entangled state defined by

$$|\alpha\rangle := \sum_m |m\rangle_s |m\rangle_e, \quad (15)$$

where $\{|m\rangle_e\}$ is a complete orthonormal set corresponding to $\{|m\rangle_s\}$. Equation (11) holds for the particular choice (15).

Proof. We rewrite the right-hand side of equation (13) as

$$\sigma(|\alpha\rangle) = (\mathcal{E} \otimes I) \left(\sum_{m,n} |m\rangle_s |m\rangle_{e_s} \langle n|_e \langle n| \right) = \sum_{m,n} |m\rangle_e \langle n| \mathcal{E}(|m\rangle_s \langle n|) \quad (16)$$

to obtain

$$e \langle m | \sigma(|\alpha\rangle) | n \rangle_e = \mathcal{E}(|m\rangle_s \langle n|). \quad (17)$$

By linearity, we arrive at the desired equation (13). \square

From the complete positivity, $\sigma(|\alpha\rangle) > 0$, for the particular case (15), we can express $\sigma(|\alpha\rangle)$ as

$$\sigma(|\alpha\rangle) = \sum_m s_m |\hat{s}_m\rangle \langle \hat{s}_m| = \sum_m |s_m\rangle \langle s_m|, \quad (18)$$

where s_m 's are positive and $\{|\hat{s}_m\rangle\}$ is a complete orthonormal set with $|s_m\rangle := \sqrt{s_m} |\hat{s}_m\rangle$. We define the Kraus operator E_m [2] as

$$E_m |\psi\rangle_s := e \langle \tilde{\psi} | s_m \rangle, \quad (19)$$

where $|\tilde{\psi}\rangle_e$ is defined in equation (14). Then, the quantum state change becomes the Kraus form

$$\begin{aligned} \sum_m E_m |\psi\rangle_s \langle\psi| E_m^\dagger &= \sum_m e \langle \tilde{\psi} | s_m \rangle \langle s_m | \tilde{\psi} \rangle_e = e \langle \tilde{\psi} | \sigma | \tilde{\psi} \rangle_e \\ &= \mathcal{E}(|\psi\rangle_s \langle\psi|). \end{aligned} \quad (20)$$

We emphasize that the quantum state change is described solely in terms of the quantities of the target system.

2.2. Quantum operations for W operators

Let us now define a W operator as

$$W(t) := |\psi(t)\rangle\langle\phi(t)|, \tag{21}$$

based on the two-state vector formalism by Aharonov and Vaidman [30] and define

$$\langle A \rangle_W := \frac{\text{Tr}(AW)}{\text{Tr}(W)} \tag{22}$$

for an observable A corresponding to the weak value of the observable A [9] as the above⁶. The weak value is an analog of a probability, and so is the W operator that of the density operators. We discuss a state change in terms of the W operator and define a map X as

$$X(|\alpha\rangle, |\beta\rangle) := (\mathcal{E} \otimes I)(|\alpha\rangle\langle\beta|), \tag{23}$$

for an arbitrary $|\alpha\rangle, |\beta\rangle \in \mathcal{H}_s \otimes \mathcal{H}_e$. Then, we obtain the following theorem on the change of the W operator such as theorem 1.

Theorem 2. For any W operator $W = |\psi(t)\rangle_s\langle\phi(t)|$, we expand

$$|\psi(t)\rangle_s = \sum_m \psi_m |\alpha_m\rangle_s, \quad |\phi(t)\rangle_s = \sum_m \phi_m |\beta_m\rangle_s, \tag{24}$$

with fixed complete orthonormal sets $\{|\alpha_m\rangle_s\}$ and $\{|\beta_m\rangle_s\}$. Then, a change of the W operator can be written as

$$\mathcal{E}(|\psi(t)\rangle_s\langle\phi(t)|) = e\langle\tilde{\psi}(t)|X(|\alpha\rangle, |\beta\rangle)|\tilde{\phi}(t)\rangle_e, \tag{25}$$

where

$$|\tilde{\psi}(t)\rangle_e = \sum_k \psi_k^* |\alpha_k\rangle_e, \quad |\tilde{\phi}(t)\rangle_e = \sum_k \phi_k^* |\beta_k\rangle_e, \tag{26}$$

and $|\alpha\rangle$ and $|\beta\rangle$ are maximally entangled states defined by

$$|\alpha\rangle := \sum_m |\alpha_m\rangle_s |\alpha_m\rangle_e, \quad |\beta\rangle := \sum_m |\beta_m\rangle_s |\beta_m\rangle_e. \tag{27}$$

Here, $\{|\alpha_m\rangle_e\}$ and $\{|\beta_m\rangle_e\}$ are complete orthonormal sets corresponding to $\{|\alpha_m\rangle_s\}$ and $\{|\beta_m\rangle_s\}$, respectively.

The proof is completely parallel to that of theorem 1.

We take the polar decomposition of the map X to obtain

$$X = \sigma u, \tag{28}$$

noting that

$$X X^\dagger = \sigma u u^\dagger \sigma = \sigma^2. \tag{29}$$

The unitary operator u is well defined on $\mathcal{H}_s \otimes \mathcal{H}_e$ because σ defined in equation (11) is positive. This is a crucial point to obtain the main result of this paper (35), which is the operator-sum representation for the quantum operation of the W operator. From equation (18), we can rewrite X as

$$\begin{aligned} X &= \sum_m |s_m\rangle\langle s_m| u \\ &= \sum_m |s_m\rangle\langle t_m|, \end{aligned} \tag{30}$$

⁶ While the original notation of the weak values is $\langle A \rangle_w$ indicating the ‘weak’ value of an observable A , our notation is motivated by one of which the pre- and post-selected states are explicitly shown as ${}_f\langle A \rangle_i^w$.

where

$$\langle t_m | = \langle s_m | u. \quad (31)$$

Similarly to the Kraus operator (19), we define the two operators, E_m and F_m^\dagger , as

$$E_m |\psi(t)\rangle_s := {}_e \langle \tilde{\psi}(t) | s_m \rangle, \quad (32)$$

$${}_s \langle \phi(t) | F_m^\dagger := \langle t_m | \tilde{\phi}(t) \rangle_e, \quad (33)$$

where $|\tilde{\psi}(t)\rangle_e$ and $|\tilde{\phi}(t)\rangle_e$ are defined in equation (26). Therefore, we obtain the change of the W operator as

$$\begin{aligned} \sum_m E_m |\psi(t)\rangle_s \langle \phi(t) | F_m^\dagger &= \sum_m {}_e \langle \tilde{\psi}(t) | s_m \rangle \langle t_m | \tilde{\phi}(t) \rangle_e = {}_e \langle \tilde{\psi}(t) | X | \tilde{\phi}(t) \rangle_e \\ &= \mathcal{E}(|\psi(t)\rangle_s \langle \phi(t) |), \end{aligned} \quad (34)$$

using theorem 2 in the last line. By linearity, we conclude

$$\mathcal{E}(W) = \sum_m E_m W F_m^\dagger, \quad (35)$$

which is one of our main results. Note that, in general, $\mathcal{E}(W)\mathcal{E}(W^\dagger) \neq \mathcal{E}(\rho)$ although $\rho = WW^\dagger$.

Summing up, we have introduced the W operator (21) and obtained the general form of the quantum operation of the W operator (35) in an analogous way to the quantum operation of the density operator assuming the complete positivity of the physical operation.

It is well established that the trace preservation, $\text{Tr}(\mathcal{E}(\rho)) = \text{Tr}\rho = 1$ for all ρ , implies that $\sum_m E_m^\dagger E_m = 1$. The proof is simple [7] and goes through as

$$1 = \text{Tr}(\mathcal{E}(\rho)) = \text{Tr} \left(\sum_m E_m \rho E_m^\dagger \right) = \text{Tr} \left(\sum_m E_m^\dagger E_m \rho \right) \quad (\forall \rho). \quad (36)$$

This argument for the density operator $\rho = WW^\dagger$ applies also for $W^\dagger W$ to obtain $\sum_m F_m^\dagger F_m = 1$ because this is the density operator in the time reversed world in the two-state vector formulation as reviewed in the introduction. Therefore, we can express the Kraus operators

$$E_m = {}_e \langle e_m | U | e_i \rangle_e, \quad F_m^\dagger = {}_e \langle e_f | V | e_m \rangle_e, \quad (37)$$

where

$$U = U(t, t_i), \quad V = U(t_f, t) \quad (38)$$

are the evolution operators, which act on $\mathcal{H}_s \otimes \mathcal{H}_e$. $|e_i\rangle$ and $|e_f\rangle$ are some basis vectors and $|e_m\rangle$ is a complete set of basis vectors with $\sum_m |e_m\rangle \langle e_m| = 1$. We can compute

$$\sum_m F_m^\dagger E_m = \sum_m {}_e \langle e_f | V | e_m \rangle_e \langle e_m | U | e_i \rangle_e = {}_e \langle e_f | V U | e_i \rangle_e. \quad (39)$$

The above equality (39) may be interpreted as a decomposition of the history in analogy to the decomposition of unity because

$${}_e \langle e_f | V U | e_i \rangle_e = {}_e \langle e_f | S | e_i \rangle_e = S_{fi} \quad (40)$$

is the S-matrix element. The meaning of the basis $|e_i\rangle$ and $|e_f\rangle$ will be clear in the following section.

As is well known [7], the physical operation for the density operator can also be described by introducing an environment which is tensored by the target system. We perform a unitary

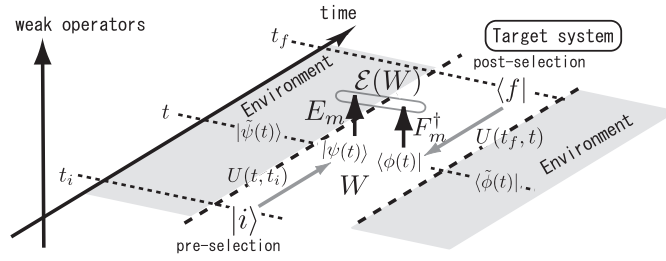


Figure 1. The quantum operations for the W operators. The W operator $W(t) = |\psi(t)\rangle_s \langle\phi(t)|$ carries the entire history from the pre-selected state to the post-selected state. The quantum operations for the W operator are described by the two operators E_m and F_m^\dagger . These operators correspond to the Kraus operators for the density operators, WW^\dagger and $W^\dagger W$, related to the two-state vector formalism and affect the history.

transformation for a combined state and then take a partial trace over the environmental states. We can also apply this method to the W operator. Namely

$$\mathcal{E}(W) = \text{Tr}_{env}[U(W \otimes e)V], \quad (41)$$

where $e = |e_i\rangle\langle e_f|$ is the environmental W operator before the physical process. It is straightforward to formally carry out the partial trace to reproduce the Kraus representation for the W operator W as equation (35). Any interaction model of this type will give the same Kraus representation for the W operator. The procedure of quantum operations for the W operator is illustrated in figure 1.

A representation of an ensemble of quantum states is usually described by a density operator since we only obtain the probability distribution of an observable in the conventional quantum measurement theory. The density operator does not contain the phase information of a quantum state. However, the W operator gives a weak value of an observable and retains the information of the phase of the quantum state. We will show a typical example of a geometric phase in section 4.

3. Weak measurement with decoherence

So far we have formally discussed the quantum operations of the W operators. In this section, we would like to study the effect of environment in the course of the weak measurement [9] and see how the shift of the probe position is affected by the environment. As we shall see, the shift is related to the quantum operation of the W operator $\mathcal{E}(W)$ (35) which we have investigated in the previous section.

3.1. Weak measurement—review

First, we recapitulate the idea of the weak measurement [9, 38]. Consider a target system and a probe defined in the Hilbert space $\mathcal{H}_s \otimes \mathcal{H}_p$. The interaction of the target system and the probe is assumed to be weak and instantaneous:

$$H_{int}(t) = g\delta(t - t_0)(A \otimes P), \quad (42)$$

where an observable A is defined in \mathcal{H}_s , while P is the momentum operator of the probe. The time evolution operator becomes

$$e^{-ig(A \otimes P)}. \quad (43)$$

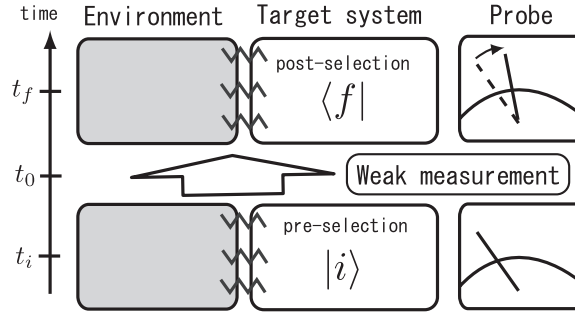


Figure 2. A weak measurement model with the environment. The environment affects the target system as a noise but does not affect the probe. The weak measurement for the target system and the probe brings about the shift of the probe position at t_0 . The amount of the shift depends on whether the environmental state is controllable (section 3.2) or uncontrollable (section 3.3).

Suppose the probe state is initially $\xi(q) \in \mathbb{R}$ in the coordinate representation with the probe position q . For the transition from the pre-selected state $|i\rangle$ to the post-selected state $|f\rangle$, the probe wavefunction becomes

$$\langle f|V e^{-ig(A \otimes P)} U|i\rangle \xi(q), \quad (44)$$

which is in the weak coupling case⁷,

$$\begin{aligned} \langle f|V[1 - ig(A \otimes P)]U|i\rangle \xi(q) &= \langle f|VU|i\rangle \xi(q) - g \langle f|VAU|i\rangle \xi'(q) \\ &\approx \langle f|VU|i\rangle \xi \left(q - g \frac{\langle f|VAU|i\rangle}{\langle f|VU|i\rangle} \right). \end{aligned} \quad (45)$$

In the previous notation, the argument of the wavefunction is shifted by

$$g \frac{\langle f|VAU|i\rangle}{\langle f|VU|i\rangle} = g \langle A \rangle_w, \quad (46)$$

so that the shift of the expectation value is the real part of the weak value, $g \cdot \text{Re}[\langle A \rangle_w]$. The shift of the momentum distribution can be similarly calculated to give $2g \cdot \text{Var}(p) \cdot \text{Im}[\langle A \rangle_w]$, where $\text{Var}(p)$ is the variance of the probe momentum before the interaction. Putting together, we can measure the weak value $\langle A \rangle_w$ by observing the shift of the expectation value of the probe both in the coordinate and momentum representations. The shift of the probe position contains the future information up to the post-selected state.

3.2. Weak measurement and environment

Let us consider a target system coupled with an environment and a general weak measurement for the compound of the target system and the environment. We assume that there is no interaction between the probe and the environment. This situation is illustrated in figure 2. The Hamiltonian for the target system and the environment is given by

$$H = H_0 \otimes I_e + H_1, \quad (47)$$

where H_0 acts on the target system \mathcal{H}_s , and the identity operator I_e is for the environment \mathcal{H}_e , while H_1 acts on $\mathcal{H}_s \otimes \mathcal{H}_e$. The evolution operators $U = U(t, t_i)$ and $V = U(t_f, t)$ as defined in equation (38) can be expressed by

$$U = U_0 K(t_0, t_i), \quad V = K(t_f, t_0) V_0, \quad (48)$$

⁷ $\xi \left(q - g \frac{\langle f|VAU|i\rangle}{\langle f|VU|i\rangle} \right)$ stands for $\xi(q)|_{q \rightarrow q - g \frac{\langle f|VAU|i\rangle}{\langle f|VU|i\rangle}}$.

where U_0 and V_0 are the evolution operators forward in time and backward in time, respectively, by the target Hamiltonian H_0 . K 's are the evolution operators in the interaction picture:

$$K(t_0, t_i) = \mathcal{T} e^{-i \int_{t_i}^{t_0} dt U_0^\dagger H_1 U_0}, \quad K(t_f, t_0) = \overline{\mathcal{T}} e^{-i \int_{t_0}^{t_f} dt V_0 H_1 V_0^\dagger}, \quad (49)$$

where \mathcal{T} and $\overline{\mathcal{T}}$ stand for the time-ordering and anti time-ordering products, respectively.

Let the initial and final environmental states be $|e_i\rangle$ and $|e_f\rangle$, respectively. The probe state now becomes

$$\langle f | \langle e_f | V U | e_i \rangle | i \rangle \xi \left(q - g \frac{\langle f | \langle e_f | V A U | e_i \rangle | i \rangle}{\langle f | \langle e_f | V U | e_i \rangle | i \rangle} \right). \quad (50)$$

Plugging the expressions for U and V into the above, we obtain the probe state as

$$N \xi \left(q - g \frac{\langle f | \langle e_f | K(t_f, t_0) V_0 A U_0 K(t_0, t_i) | e_i \rangle | i \rangle}{N} \right), \quad (51)$$

where $N = \langle f | \langle e_f | K(t_f, t_0) V_0 U_0 K(t_0, t_i) | e_i \rangle | i \rangle$ is the normalization factor. We define the dual quantum operation as

$$\begin{aligned} \mathcal{E}^*(A) &:= \langle e_f | K(t_f, t_0) V_0 A U_0 K(t_0, t_i) | e_i \rangle \\ &= \sum_m V_0 F_m^\dagger A E_m U_0, \end{aligned} \quad (52)$$

where

$$F_m^\dagger := V_0^\dagger \langle e_f | K(t_f, t_0) | e_m \rangle V_0, \quad E_m := U_0 \langle e_m | K(t_0, t_i) | e_i \rangle U_0^\dagger \quad (53)$$

are the Kraus operators introduced in the previous section (37). Here, we have inserted the completeness relation $\sum_m |e_m\rangle \langle e_m| = 1$ with $|e_m\rangle$ being not necessarily orthogonal. The basis $|e_i\rangle$ and $|e_f\rangle$ are the initial and final environmental states, respectively. This provides the meaning of $|e_i\rangle$ and $|e_f\rangle$ as alluded before. Thus, we obtain the wavefunction of the probe as

$$\begin{aligned} \xi \left(q - g \frac{\langle f | \mathcal{E}^*(A) | i \rangle}{N} \right) &= \xi \left(q - g \frac{\sum_m \langle f | V_0 F_m^\dagger A E_m U_0 | i \rangle}{\sum_m \langle f | V_0 F_m^\dagger E_m U_0 | i \rangle} \right) \\ &= \xi \left(q - g \frac{\text{Tr}[A \sum_m E_m U_0 | i \rangle \langle f | V_0 F_m^\dagger]}{\text{Tr}[\sum_m E_m U_0 | i \rangle \langle f | V_0 F_m^\dagger]} \right) \\ &= \xi \left(q - g \frac{\text{Tr}[\mathcal{E}(W)A]}{\text{Tr}[\mathcal{E}(W)]} \right) = \xi(q - g \langle A \rangle_{\mathcal{E}(W)}), \end{aligned} \quad (54)$$

with $N = \langle f | \mathcal{E}^*(I) | i \rangle$ up to the overall normalization factor. This is the main result of this subsection. The shift of the expectation value of the position operator on the probe is

$$\delta q = g \cdot \text{Re}[\langle A \rangle_{\mathcal{E}(W)}]. \quad (55)$$

From an analogous discussion, we obtain the shift of the expectation value of the momentum operator on the probe as

$$\delta p = 2g \cdot \text{Var}(p) \cdot \text{Im}[\langle A \rangle_{\mathcal{E}(W)}]. \quad (56)$$

Thus, we have shown that the probe shift in the weak measurement is exactly given by the weak value defined by the quantum operation of the W operator due to the environment, which is one of our main results.

3.3. Weak measurement—decoherence

In many cases, the initial and final states, e_i and e_f , of the environment, on which the quantum operation \mathcal{E} depends, are not controllable so that they have to be statistically treated. Let the statistical weight be $w(e_f, e_i)$ and consider the average

$$\text{Ave}(h) := \sum_{e_f, e_i} w(e_f, e_i) h(e_f, e_i), \quad (57)$$

for a function h of the random variables e_f and e_i . We proceed with

$$\begin{aligned} \text{Ave} \left(N \xi \left(q - g \frac{\text{Tr}[\mathcal{E}(W)A]}{N} \right) \right) &\approx \text{Ave} \left(N \left\{ \xi(q) - g \frac{\text{Tr}[\mathcal{E}(W)A]}{N} \xi'(q) \right\} \right) \\ &= \text{Ave}(N) \left[\xi(q) - g \frac{\text{Ave}[\text{Tr}\mathcal{E}(W)A]}{\text{Ave}(N)} \xi'(q) \right] \\ &\approx \text{Ave}(\text{Tr}\mathcal{E}(W)) \xi \left(q - g \frac{\text{Ave}(\text{Tr}[\mathcal{E}(W)A])}{\text{Ave}(\text{Tr}[\mathcal{E}(W)])} \right), \end{aligned} \quad (58)$$

noting that

$$\begin{aligned} N &= \langle f | \mathcal{E}^*(I) | i \rangle = \sum_m \langle f | V_0 F_m^\dagger E_m U_0 | i \rangle \\ &= \sum_m \text{Tr} E_m U_0 | i \rangle \langle f | V_0 F_m^\dagger = \text{Tr} \mathcal{E}(W). \end{aligned} \quad (59)$$

We see that the shift of the expectation value of the probe position is on average

$$\delta q = \text{Re} \left[g \frac{\text{Ave}(\text{Tr}[\mathcal{E}(W)A])}{\text{Ave}(\text{Tr}[\mathcal{E}(W)])} \right], \quad (60)$$

in the weak coupling case. To obtain a significant shift, one needs some prior knowledge of the environment. For the case of a detector as the environment, e_i and e_f are specified by the measurement outcome and are definite if the environment is at zero temperature, for example.

4. Geometric phase

We present a simple application of our framework of the physical operation of the W operators to the geometric phase since measuring the geometric phase is equivalent to measuring the weak value in the qubit system [41]. There have been many works on the ‘geometric phase for mixed states’ but the very definition seems under controversy [42–48]. We would like to start with the geometric phase γ in a pure state [39] which is well defined and can be expressed in terms of the weak value

$$\gamma = \arg \left[\frac{\text{Tr}(WP)}{\text{Tr}(W)} \right], \quad (61)$$

where $P = |P\rangle\langle P|$ is a projector to a pure state and $W = |\tilde{i}\rangle\langle\tilde{f}|$. Here, we simplify the notations $|\tilde{i}\rangle := U(t, t_i)|i\rangle$ and $\langle\tilde{f}| := \langle f|U(t_f, t)$ only in this section. The geometric phase γ corresponds to the quantum path $|\tilde{i}\rangle \rightarrow |P\rangle \rightarrow |\tilde{f}\rangle$. By the physical operation \mathcal{E} , the W operator and the density operator are mapped to $\mathcal{E}(W)$ and $\mathcal{E}(\rho)$, respectively. The new state $\mathcal{E}(\rho)$ is in general a mixed state. The new geometric phase γ_g is correspondingly given by

$$\gamma_g = \arg \left[\frac{\text{Tr}(\mathcal{E}(W)P)}{\text{Tr}\mathcal{E}(W)} \right], \quad (62)$$

which might be called the geometric phase of the mixed state $\mathcal{E}(\rho)$ by a slight abuse of words. This operational definition fits well to experimental situation. That is, an experimentalist starts

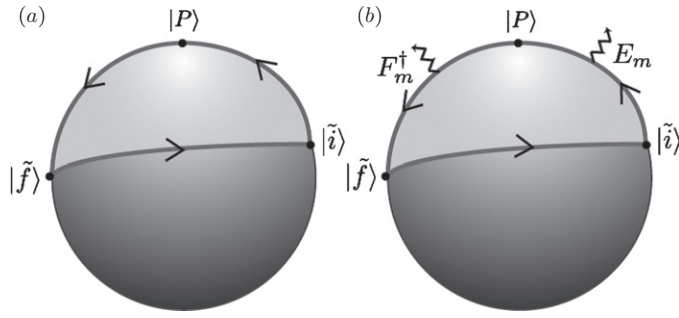


Figure 3. (a) Geometric phase in the case of the pure state for a qubit system. When the path $|\tilde{i}\rangle \rightarrow |P\rangle \rightarrow |\tilde{j}\rangle$, the half of the solid angle on the Bloch sphere, which is the gray region, corresponds to the geometric phase. (b) Geometric phase in the presence of the environment with the same path. The quantum operations are represented by E_m and F_m^\dagger .

with a pure state $|\tilde{i}\rangle$ and then makes a trip $|\tilde{i}\rangle \rightarrow |P\rangle \rightarrow |\tilde{j}\rangle$ by manipulating the external field. If there were no decoherence during that process, one would get the geometric phase defined above (61). Otherwise, one would instead get the value (62) for the geometric phase, while one can presume that the state is $\mathcal{E}(\rho) = \mathcal{E}(WW^\dagger)$. Furthermore, we would like to point out that this definition (62) coincides with the formal definition in the Uhlmann approach [36, 48]. In the generalized Kraus representation, the geometric phase for that path, $|\tilde{i}\rangle \rightarrow |P\rangle \rightarrow |\tilde{j}\rangle$, can be written as

$$\gamma_g = \arg \left[\frac{\sum_m \langle \tilde{j} | F_m^\dagger | P \rangle \langle P | E_m | \tilde{i} \rangle}{\sum_m \langle \tilde{j} | F_m^\dagger E_m | \tilde{i} \rangle} \right]. \tag{63}$$

This is illustrated in figure 3. The expression (63) is operationally defined from the view of the decoherence and may be regarded as an improvement of equation (10) in the paper [48].

Let us see the decoherence effect on the geometric phase in a simple one qubit system under a bit flip noise [7]. The Kraus operators are chosen by

$$E_0 = F_0 = \sqrt{p}I, \quad E_1 = F_1 = \sqrt{1-p}\sigma_x, \tag{64}$$

with $0 \leq p \leq 1$; note that

$$\sum_m F_m^\dagger E_m = I \tag{65}$$

is the unitary operator and $\sum_m E_m^\dagger E_m = \sum_m F_m^\dagger F_m = I$. We are going to consider a path, $|\tilde{i}\rangle \rightarrow |0\rangle \rightarrow |\tilde{j}\rangle$, where

$$|\tilde{i}\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle), \quad |\tilde{j}\rangle = \frac{1}{\sqrt{2}}(|0\rangle - e^{-i\phi}|1\rangle), \tag{66}$$

where $|\tilde{j}\rangle$ is a rotated state from $|\tilde{i}\rangle$ by the angle ϕ . A straightforward calculation shows

$$\langle P_0 \rangle_{\mathcal{E}(W)} = \frac{p + (1-p)e^{i\phi}}{1 + e^{i\phi}} \tag{67}$$

for the projection $P_0 = |0\rangle\langle 0|$. Then, the geometric phase γ_g (63) is

$$\gamma_g = \arg \langle P_0 \rangle_{\mathcal{E}(W)} = \arctan \left[(1-2p) \tan \frac{\phi}{2} \right]. \tag{68}$$

In the no noise case, $p = 1$, we recover the geometric phase $-\phi/2$, which is the half of the solid angle formed by the three vectors $|\tilde{i}\rangle$, $|0\rangle$ and $|\tilde{j}\rangle$ in the Bloch sphere. In the case of $p = 1/2$, the geometric phase vanishes as we expect because the state is completely mixed. It seems that the decohered geometric phase has no particular geometrical meaning while Uhlmann gave an expression for the geometric phase in terms of operators similar to the W operator [49, 50]. It is curious to point out that the geometric phase during the measuring process is given by a time evolution of weak values [51].

We would like to stress that our definition of the geometric phase under the environmental noise is operationally defined in the sense that the geometric phase is initially defined in a pure state but undergoes a decoherence process while the state becomes a mixed state. Hence, the geometric phase of the mixed state (62) can be experimentally verifiable.

5. Summary

We have introduced the W operator W (5) to formally describe the weak value advocated by Aharonov *et al* which is the more general quantity containing the phase information than the density operator ρ . The general framework is given to describe the effects of quantum operation $\mathcal{E}(W)$ to the W operator W in parallel with the Kraus representation of the CP map for the density operator ρ . The result is the change of the history,

$$\mathcal{E}(W) = \sum_m E_m W F_m^\dagger, \quad (69)$$

with $\sum_m E_m^\dagger E_m = \sum_m F_m^\dagger F_m = 1$ and $\sum_m F_m^\dagger E_m = S_{fi}$ is the S-matrix element corresponding to the history (see figure 1). We have shown the effect of the environment during the weak measurement as the shift of the expectation value of the probe observables in both cases of controllable and uncontrollable environmental states. As an application, it can demonstrate how the geometric phase is affected by decoherence, which is experimentally testable.

Extending our proposed definition of the W operators, we may consider a superposition of W operators

$$W := \sum_{i,f} \alpha_{if} U(t, t_i) |i\rangle \langle f| U(t_f, t), \quad (70)$$

in analogy to the mixed state which is a convex linear combination of pure states. Actually, $\mathcal{E}(W)$ (35) has the form (70). Although this indicates a time-like correlations, the physical implication is not yet clear. This operator may be related to the concept of the multi-time states [52]. In fact, it is shown how the weak value corresponding to the W operator (70) can be constructed via a protocol by introducing auxiliary states which are space-likely entangled with the target states.

Acknowledgments

We would like to acknowledge useful comments with Dr Alberto Carlini, Professor Masanao Ozawa and Professor Holger Hofmann and one of the authors (YS) would like to also acknowledge useful discussions with Professor Masao Kitano. We would like to thank the support from Global Center of Excellence Program ‘Nanoscience and Quantum Physics’ at Tokyo Institute of Technology. YS acknowledges JSPS Research Fellowships for Young Scientists (grant no 21008624).

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